Nonconservative Abelian sandpile model with the Bak-Tang-Wiesenfeld toppling rule

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A nonconservative Abelian sandpile model with the Bah-Tang-Wiesenfeld toppling rule introduced by Tsuchiya and Katori [Phys. Rev. E **61**, 1183 (2000)] is studied. Using a scaling analysis of the different energy scales involved in the model and numerical simulations it is shown that this model belongs to a universality class different from that of previous models considered in the literature.

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I. INTRODUCTION

Recently Tsuchiya and Katori [1] have introduced a nonconservative Abelian sandpile model with a toppling rule similar to that of the well known Bak, Tang, and Wiesenfeld (BTW) sandpile model [2]. The model is defined in a square lattice of size L and an integer energy profile z_{ij} is considered. Sites with energy below the threshold $z_c=4\alpha\zeta$ are stable. If the energy at any given site (i,j) exceeds this threshold the site transfers energy to its four nearest neighbors following the toppling rule: $z_{ij}\to z_{ij}-z_c$, $z_{i\pm 1j}\to z_{i\pm 1j}+\zeta$ and $z_{ij\pm 1}\to z_{ij\pm 1}+\zeta$. ζ is an integer number and α is such that z_c is also an integer. The boundaries are assumed to be open and the system is perturbed by adding a unit of energy at a site selected at random and letting it evolve until an equilibrium configuration is reached.

On each toppling event an amount of energy $\epsilon=4\zeta(\alpha-1)>0$ is dissipated. For $\alpha=1$ the model is conservative; it is just the BTW model but with a different scale of toppling and energy addition. In the BTW model [2] the energy added to perturb the system is 1 and on toppling an active site transfers an energy 1 to each neighbor; they are of the same order. In the model defined above the energy added is still 1 but the energy transferred on toppling is ζ. In the limit $\alpha=1$ and $\zeta=1$ the BTW model [2] is recovered while for $\alpha=1$ and $\zeta\gg 1$ it is similar to the BTW model but with a uniform driving.

In the BTW model ($\alpha=1$ and $\zeta=1$) the avalanches can be decomposed in a sequence of subavalanches called waves [3] with well-defined finite-size scaling properties. On the contrary, the distribution of the overall avalanche size s is better described using a multifractal analysis [4]. The break down of the finite-size scaling has been recently shown to be a consequence of the existence of correlations in the sequence of waves [5].

For $\alpha>1$ the model is nonconservative but still Abelian [1]. In the thermodynamic limit $L\to\infty$, exact calculations by Tsuchiya and Katori yield the mean avalanche size (including avalanches with size zero) $\langle T\rangle=\epsilon^{-1}$ [1]. Since $\epsilon=4\zeta(\alpha-1)$ they concluded that in the limit $\alpha\to 1\langle T\rangle$ diverges. However, as it is shown in Sec. II this conclusion is wrong because α cannot go to zero in an arbitrary way, in order to satisfy the constraint that $z_c=4\alpha\zeta$ remains integer. Here it is demonstrated that $\epsilon\geq 1$ and, therefore, $\langle T\rangle\leq 1$ for all possible values of ζ and α.

The main goal of this work is to investigate the scaling properties of this nonconservative BTW-like model in the limit $\alpha\to 1$. The main questions are related to the existence or not of criticality in the conservative limit $\alpha\to 1$ and if in this limit one recovers the conventional BTW model $\alpha=1$. From the analysis of the energy scales involved in the model (Sec. II) and numerical simulations (Sec. III) it is concluded that the model is critical when $\alpha\to 1$ but it does not belong to the universality class of the BTW model. Its relation to other nonconservative models with the BTW-like toppling rule introduced in the literature [6–8] is also discussed (Sec. III).

II. SCALING LAWS

In this section some scaling laws are derived based on the energy scales involved in the model. The main idea of this approach is that the balance between input and dissipation of energy determines the scaling of some magnitudes with the dissipation per toppling event, following the general guidelines introduced by Vespiganani *et al.* [9]. For this purpose the avalanches are assumed to be instantaneous and the analysis is focused on the time scale of the driving field. On each step one adds 1 unit of energy and measures the toppling activity and the energy dissipated. On each toppling event an amount of energy $\epsilon=4\zeta(\alpha-1)$ is locally dissipated while an amount 4ζ is transferred to nearest neighbors. For boundary sites part of the energy is also dissipated through the boundary.

Let $G(r)$ be the Green function [10], the average number of toppling events at a distance r from the site where the energy was added. Close to $r=0$ the effect of local dissipation gives a small contribution and the main energy scale is given by the transport of the energy from active sites to their nearest neighbors. On the contrary, far from $r=0$ the effects of the local dissipation becomes more important. How far will depend on the certain correlation length ξ, such that for $r\ll\xi$ transport is more important than local dissipation while for $r\gg\xi$ the opposite occurs.

Thus, there are two characteristic lengths in this model: the system size L and the correlation length ξ. The analysis developed above is valid in the thermodynamic limit $L\gg\xi$. In this case the dissipation through the boundary of the system is negligible in comparison with the energy dissipated on each toppling event. In such a situation the only way to reach a stationary state is to balance the input of energy from the driving field with the energy locally dissipated. Moreover, since ξ is the only characteristic length $G(r)$ is expected to satisfy the scaling law

$$G(r)=r^{\eta-d}\mathcal{F}(r/\xi), \tag{1}$$

where η is an scaling exponent and d is the spatial dimension.

The amount of energy $\delta E_d(r)$ locally dissipated inside a hypercircle of radius r is

$$\delta E_d(r)\propto\epsilon\int_0^r d\rho\rho^{d-1}G(\rho)\propto\epsilon\xi^{\eta}f(r/\xi), \tag{2}$$

where $f(x)=\int_0^x dyy^{\eta-1}\mathcal{F}(y)$ and the second proportionality is obtained using Eq. (1). On the other hand, the average energy transported through its boundary $\delta E_t(r)$ is given by

$$\delta E_t(r)\propto\zeta r^{d-1}\frac{dG}{dr}(r)\propto\zeta\xi^{\eta-2}g(r/\xi), \tag{3}$$

where $g(x)=d[x^{\eta-2}\mathcal{F}(x)]/dx$ and the second proportionality is obtained using Eq. (1). The correlation length ξ can be defined as the radius r at which these two contributions become of the same order. With this definition and equating Eqs. (2) and (3) with $r=\xi$ it results that

$$\xi\sim\left(\frac{\zeta}{\epsilon}\right)^{\nu}, \quad \nu=\frac{1}{2}. \tag{4}$$

On the other hand, on each step 1 unit of energy is added and on average the amount $\epsilon\langle T\rangle$ is dissipated, where $\langle T\rangle\propto\int_0^\infty drr^{d-1}G(r)$ is the mean avalanche size, including avalanches with size 0. Equating these two contributions it results that

$$\langle T\rangle=\frac{1}{\epsilon}=\frac{1}{4\zeta(\alpha-1)}. \tag{5}$$

Moreover, using Eq. (1) one obtains

$$\eta=0. \tag{6}$$

Equation (5) reproduces the exact result by Tsuchiya and Katori. The present approach is, however, based on more general arguments and can be easily adapted to any sandpile model with local dissipation. The same argument (energy balance) has been previously used by Vespiganani *et al.* [9] to understand the scaling properties of other sandpile models with local dissipation. Here, a slightly different approach has been considered where the new parameter ζ, the ratio between the energy received from nearest active neighbors and from the driving field, has been taken into account.

From Eq. (5) Tsuchiya and Katori concluded that when $\alpha\rightarrow1\langle T\rangle$ diverges. However, this conclusion is not valid if $z_c=4\zeta\alpha$ is restricted to be an integer number. To show this let us write $\alpha=1+\epsilon/4\zeta$ which follows from Eq. (5). But $4\zeta\alpha=4\zeta+\epsilon$ is restricted to take integer values. With ζ being an integer number the only way to satisfy this requirement is that ϵ is also integer, i.e., $\epsilon=1,2,3,\ldots$. Then, since the smaller non-negative integer is 1 it is concluded that $\epsilon\geqslant1$ and, therefore, from Eq. (5) $\langle T\rangle\leqslant1$, i.e., it is bounded.

Nevertheless, the correlation length ξ in Eq. (4) does not only depend on ϵ but also on ζ. For fixed ϵ it diverges in the limit $\zeta\rightarrow\infty$ and the model is critical. The real control parameter is then $\epsilon_{eff}=\epsilon/\zeta$, i.e., the energy dissipated per toppling event relative to the characteristic energy scale of transport ζ. Although this result is in complete agreement with the field theory approach of Vespignani *et al.* [9] the fact that $\langle T\rangle$ does not diverge when $\epsilon_{eff}\rightarrow0$ ($\zeta\rightarrow\infty$) excludes this model from their analysis.

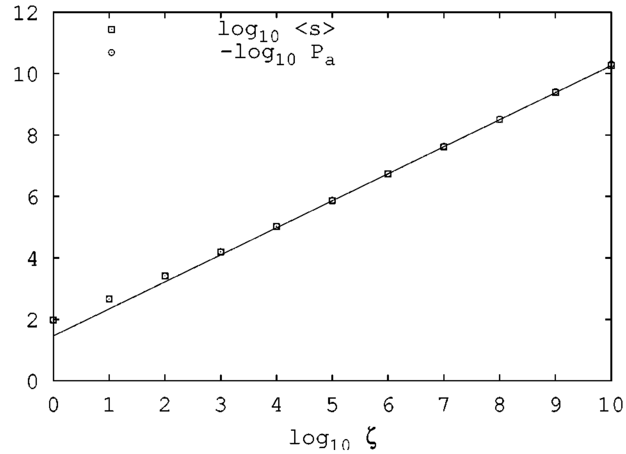

FIG. 1. Log-log plot of the mean avalanche size (excluding avalanches with size $s=0$) as a function ζ. It can be clearly seen that it scales as P_a^{-1}, the probability per unit step to obtain an avalanche with $s>0$. The line is a linear fit to the high ζ interval.

Moreover, in previous sandpile models conservation implies the scaling law $\langle s\rangle\sim\xi^2$, where $\langle s\rangle$ is the mean avalanche size excluding those with size 0 [9]. To investigate the validity of such a scaling relation for the present model let us take into account that $\langle s\rangle$ is related to $\langle T\rangle$ through the expression

$$\langle s\rangle=\langle T\rangle/P_a, \tag{7}$$

where P_a is the probability of obtaining an avalanche with nonzero size. In the models considered by Vespignani *et al.* [9] $\zeta=1$ and, therefore, from Eqs. (4), (5), and (7) it results that $\langle s\rangle\sim\xi^2/P_a$. Moreover, in this model P_a has a finite value and, therefore, one obtains the mentioned scaling law $\langle s\rangle\sim\xi^2$.

On the contrary, in the model considered here $\langle s\rangle$ cannot be related to ξ using these arguments. For fixed ϵ from Eqs. (5) and (7) one obtains that $\langle s\rangle\sim1/P_a$. Thus, from the energy balance invoked above we cannot say anything about the scaling of $\langle s\rangle$ with ξ (an exponent 2 will be an accidental coincidence) and, therefore, this model belongs to a new universality class.

III. NUMERICAL SIMULATIONS AND DISCUSSION

In this section results obtained from numerical simulations of the model studied above are presented. The simulations were performed using $\epsilon=1$, $L=4096$, and ζ ranging from 2^0 to 2^{10} ($\epsilon_{eff}=1/\zeta$ ranging from 1 to 2^{-10}). For these values the condition $L\ll\xi$ was observed to be satisfied. Statistics was taken over 10^8 avalanches after the system reached the stationary state.

Before entering in the analysis of the statistics of the avalanches let us check the validity of Eq. (5). The log-log plot of $\langle s\rangle$ vs ζ is shown in Fig. 1. A clear linear behavior is observed for $\log_{10}\zeta\geqslant5$ suggesting that above this value

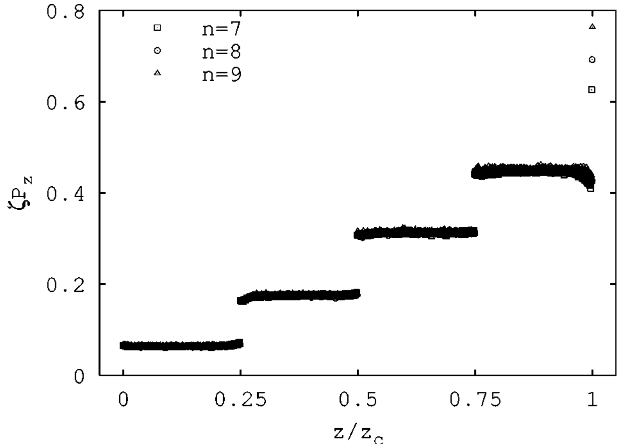

FIG. 2. Probability P_z that a site has energy z in the stationary state for different values of $\zeta = 2^n$. z is expressed in units of the threshold $z_c = 4\zeta + 1$ while P_z has been rescaled by an amount ζ because with increasing ζ the density of z/z_c values increase as ζ.

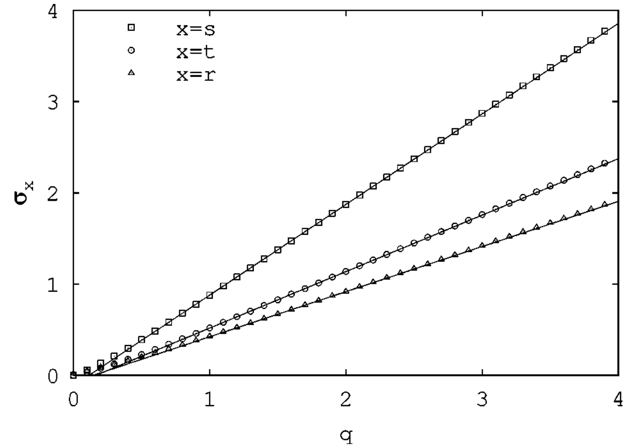

FIG. 3. Moment exponent $\sigma_x(q)$ for different values of q and $x = s,t,r$. The lines are linear fits $[\sigma_x(q) = (1-\tau_x)d_x + d_x q]$ to the interval $1 \geqslant q \leqslant 3$. The resulting exponents τ_x and d_x are shown in Table I.

simple scaling applies. On top of these points the numerically computed values of $1/P_a$ are plotted obtaining an overlap in agreement with Eq. (5). If the scaling relation $\langle s \rangle \sim \xi^2$ were valid, using Eq. (4), $\langle s \rangle \sim \zeta$. However, a linear fit to this log-log plot gives a slope ~ 0.9.

The fact that this scaling relation does not hold is clearly shown in Fig. 2, where the stationary energy distribution is shown. As can be seen, $\zeta P_a = \zeta P_{\{z_c - 1\}}$ increases with increasing ζ and, therefore, P_a decreases more slowly than $1/\zeta$. The rest of the distribution scales like $1/\zeta$ which is just a consequence of the increase of the density of possible values of z.

The avalanche statistics will be characterized by the number of toppling events s and steps t required to reach a stable configuration, the number of sites a ''touched'' by the avalanche, and the characteristic radius of the cluster formed by these sites r. The main goal of the simulations is to determine the probability densities $p_x(x,\zeta)$ $(x = s,t,a,r)$ in the stationary state.

One can easily see that $s = a$; in other words sites topple only once within an avalanche. In this model, as a difference with the original BTW model, only one wave of topplings takes place. The first wave is generated from an initial site with height $z = z_c = 4\zeta + \epsilon$. When this site topples it transfers an amount equal to z_c to its nearest neighbors and, therefore, ends with energy $z = 0$. The best we can have to obtain a second toppling at this site is that its four nearest neighbors also become active. In such a case the initial side will receive $4\zeta < z_c$ units of energy, which is not enough to make it active again. Hence, no second wave will be obtained yielding $s = a$.

Since the waves are known to satisfy well-defined finite-size scaling properties and in the present model an avalanche is made by one wave, it is expected that the distributions $p_x(x,\zeta)$ also satisfy a finite-size scaling. However, the scaling exponents will not necessarily be those obtained for the scaling of waves because, in the present model, conservation does not introduce any scaling relation among the different scaling exponents.

If finite-size scaling applies then these densities will satisfy

$$p_x(x,\zeta) = x^{-\tau_x} \mathcal{G}[x/x_c(\zeta)], \tag{8}$$

where τ_x is the power-law exponent characterizing the self-similar regime and x_c is a cutoff above which the distribution deviates from a power law and has a fast decay given by $\mathcal{G}$. The validity of this scaling form is supported by the numerical results. The cutoff x_c is determined by the existence of the characteristic length $\xi \sim \zeta^\nu$ and is expected to scale as $x_c \sim \xi^{D_x} \sim \zeta^{d_x}$, where $d_x = D_x \nu$ is an effective fractal dimension.

To compute the exponents τ_x and d_x the moment analysis technique introduced by De Menech *et al.* [11] is used. The moments of the probability density in Eq. (8) are given by

$$\langle x^q \rangle = \int_0^\infty dx p(x) x^q \sim \zeta^{\sigma_x(q)}, \tag{9}$$

where

$$\sigma_x(q) = (1-\tau_x)d_x + d_x q. \tag{10}$$

The last equivalence in Eq. (9) is valid for values of q not too small, for which the precise form of $p_x(x,\zeta)$ at small x is not important.

$\sigma_x(q)$ can be determined from a linear fit to the log-log plot of $\langle x^q \rangle$ vs ζ. The resulting values using $\zeta = 2^5, 2^6, \ldots, 2^{10}$ are shown in Fig. 3. In all cases ($x = s,t,r$) for q larger than 1 a well-defined linear dependence is observed. From the linear fit [see Eq. (10)] to these plots the exponents τ_x and d_x are computed. The results are shown in Table I.

The exponent ν is very close to 1/2 in very good agreement with the scaling arguments of preceding section. On the

TABLE I. Scaling exponents obtained from linear fits $[\sigma_x(q) = (1-\tau_x)d_x + d_x q]$ to the data shown in Fig. 3.

d_s	d_t	$d_r = \nu$	τ_s	τ_t	τ_r
0.994(5)	0.630(5)	0.495(5)	1.11(1)	1.16(1)	1.14(1)
$D_s = d_s/\nu$	$z = D_t = d_t/\nu$				
2.01(1)	1.27(1)				

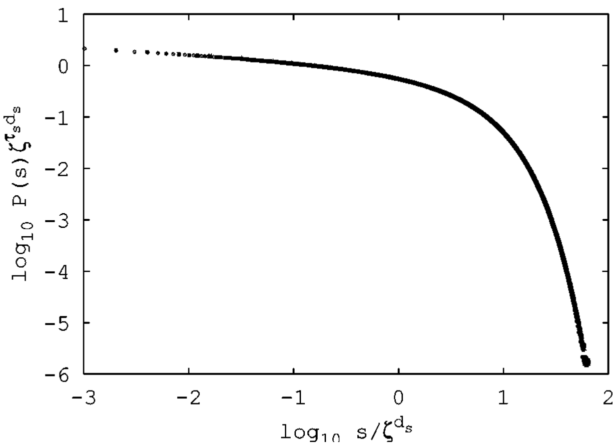

FIG. 4. Scaled plot of the integrated distribution of avalanche sizes (or area since $s=a$ in this model) using the exponents displayed in Table I.

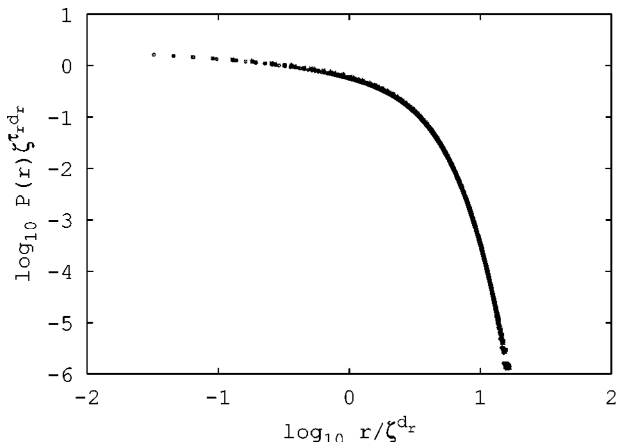

FIG. 6. Scaled plot of the integrated distribution of the avalanche radius using the exponents displayed in Table I.

other hand, d_s is quite close to 1 which implies that the avalanche size (or area) scale as $s\sim r^2$, i.e., avalanches are compact $D_s=2$. With this value, the scaling relation $(2-\tau_s)D_s=2$ yields the power-law exponent $\tau_s=1$ which is clearly in disagreement with the value computed numerically. The reason for this result is that conservation does not introduce any scaling relation as it generally occurs in sandpile models [12].

The exponents computed using the moment analysis technique can be checked using rescaled plots of the integrated distribution $P_x(x,\zeta)=\int_x^\infty dx p_x(x,\zeta)$. The resulting plots are shown in Figs. 4–6. The scaling works quite well supporting the validity of the reported exponents.

In the literature we can find other sandpile models with local dissipation and the BTW-like toppling rule [6–8]. In the models considered in [6] and [7] the energy profile is continuous and the dissipation rate per toppling event ϵ is a control parameter that can take any real value and, therefore, can be tuned to zero. Another feature of these models is that only one wave of toppling can take place and, therefore, for any finite ϵ the model is in a different universality class from that of the BTW model.

On the other hand, in [8] the energy profile is discrete as in the original BTW model at the prize of introducing stochasticity in the model. In this case with a probability p energy is fully dissipated yielding an average dissipation per toppling event $\epsilon=2dp$. Clearly p may take any real variable between 0 and 1 and, therefore, also in this case the dissipation per toppling event can be fine tuned to zero. As a difference with the models described in the previous paragraph, in this case multiple toppling of a site within an avalanche is possible, which make it closer to the original BTW model. Moreover, the use of finite-size scaling techniques can be also questioned and a multifractal analysis may be more appropriate [13], which is another characteristic feature of the BTW model [4]. All these elements together with the numerical results reported in [8] suggest that in the limit $p\to 1$ $(\epsilon\to 0)$ this model belongs to the same universality class of the BTW model.

A common feature of all these models [6–8] is that $\langle s\rangle\sim\epsilon^{-1}$ as predicted by the field theory approach of Vespignani *et al.* [9], leading to the scaling relation $(2-\tau_s)D_s=2$. On the contrary, in the present model the scaling of $\langle s\rangle$ with ϵ_{eff} is not known and conservation does not introduce the above scaling relation. Hence, the model introduced by Tsuchiya and Katori belongs to different class among sandpile models.

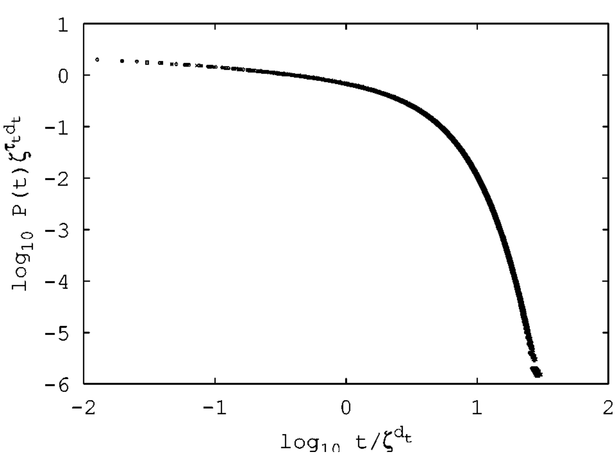

FIG. 5. Scaled plot of the integrated distribution of avalanche durations using the exponents displayed in Table I.

IV. SUMMARY AND CONCLUSIONS

A nonconservative Abelian sandpile model with a BTW-like toppling rule has been studied. The model can be thought of as the only possible generalization of the BTW model to include local dissipation without introducing stochasticity in the toppling rule and keeping a discrete energy profile. However, the scaling approach and the numerical simulations reported here show that it does not belong to the universality class of the BTW model, not even to the universality class of any sandpile model previously considered in the literature.

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